A Note on Measurement of Utility

THE possibility of measuring the marginal utility of income from budgetary studies and market behaviour has been investigated by many writers.[1] In this note a possible alternative method, resting upon different assumptions, is presented, not so much in the hope of furthering inductive investigation in these matters as of bringing out certain theoretical relations between the variables under consideration.

In order to arrive inductively at the measurement of utility, essentially a subjective quantity, it is necessary to place the individual (*homo economicus*) whose scale is sought under certain ideal circumstances where his observable behaviour will render open to *unambiguous* inference the form of the function which he is conceived of as maximising.

As has long been known, the behaviour of an individual in the market, when confronted with various price combinations and under limitations of various incomes, is not in itself sufficient to determine the form of his utility function, but can only give us at best a system of indifference loci to which an infinite number of utility indexes might have given rise, integrability conditions being met.[2]

Under the following four assumptions, it is believed possible to arrive theoretically at a precise measure of the marginal utility of *money income* to an individual whose tastes maintain a certain invariance throughout the time under consideration, and during which time the prices of all goods remain constant.[3]

[1] Cf. Irving Fisher, . . . *Econ. Essays in Honor of J. B. Clark*; R. Frisch, *New Methods of Measuring Marginal Utility*.

[2] To arrive at a unique measure of utility certain extra assumptions must be made in order to select a particular scale out of the infinitude of possible scales. We might assume, for example, that our utility function must be an additive function of the utilities corresponding to each commodity. This in turn involves an axiom (definition) and an hypothesis. The hypothesis, subject to refutation by the statistical data, is that such an additive function could possibly have given rise to the observable behaviour. The other assumption consists in *defining* that additive function, subject to freedom of a scale constant and choice of origin, as *the* utility function, in preference to any monotonic function of this additive function. This definition, like any other, is axiomatic and not subject to verification or refutation.

The choice of such an additive function can hardly be defended as a first approximation. The fact that any multiple valued function may be expanded in a Taylor series, certain terms of which form an additive function in the many variables, does not justify our assuming such a function to fit the data over an extended range, any more than we may fit a straight line to a single valued function because in the neighbourhood of any point the function may be approximated by its tangent. Moreover, if we consider the possible uses or mis-uses to which our results might be put, it is likely that we may be interested in the very properties of the function, from which we have already abstracted, else our original indifference field would have sufficed.

[3] The marginal utility of income derived will hold only for a given set of prices. This limitation is inherent in the problem, since the concept has meaning only with reference to a fixed set of prices. Moreover, we could have easily considered the expenditure on each commodity over time, and thus have solved for the marginal utility of each commodity as a function of all the variables without making any assumptions as to independence. Under assumptions analogous to the usual integrability conditions, we could then go directly to the utility function in terms of all commodities. However, this would have added to the elaborateness of our equations without introducing any results of additional analytical interest.

It may be noted here, that marginal utility of money income as here defined will *not* in general be identical with marginal utility of money income as defined by other writers.

1. *Utility is uniquely measurable as, in consequence, is marginal utility.* A fixed set of prices being given, variations of income will define a path along which we may measure the marginal utility of money income. It is the form of this function which we are trying to determine. Utility (U) of income (x) is regarded as a time flow, a rate per unit-time. Marginal (degree of) utility is the rate of utility per dollar (X), per unit of time with dimension ($UX^{-1}T^{-1}$). Mathematically

$$U = U(x) \quad \ldots\ldots \quad (1)$$

where x representing money income per unit-time is of dimension (XT^{-1}).

2. *During any specified period of time, the individual behaves so as to maximise the sum of all future utilities, they being reduced to comparable magnitudes by suitable time discounting.* This is in the nature of an axiom, or definition, not subject to proof in any empirical sense, since any and all types of observable behaviour might conceivably result from such an assumption. Mathematically, the following integral is to be made a maximum subject to side conditions to be imposed later:

$$J = \int_0^b V(x, t)dt \quad \ldots\ldots \quad (2)$$

where the beginning of the time period under consideration is taken as the origin along the time scale, and where b represents the end of this time period, here taken as finite.

3. *The individual discounts future utilities in some simple regular fashion which is known to us.* For simplicity, we assume in the first instance that the rate of discount of future utilities is a constant. This constant might of course be such that there is no time preference whatsoever, or even a premium on future utilities. This third assumption, unlike the previous two, is in the nature of an hypothesis, subject to refutation by the observable facts, i.e. refutation in the sense of proved inconsistency with the previous axioms. Moreover, this assumption, added to the previous ones, serves to limit our marginal utility function to a sub-class of all possible functions, from which sub-class it will be possible to identify a unique utility function. The arbitrariness of these assumptions is again stressed mathematically:

$$V(x, t) = U(x)e^{-\pi t} \quad \ldots\ldots \quad (3)$$

where π bears the following familiar relationship to the rate of discount (positive or negative) p, here assumed to be constant:

$$\pi = \log e(1+p) \quad \ldots\ldots \quad (3.1)$$

In accordance with this assumption we may re-write equation (2) as follows:

$$J = \int_0^b U(x)e^{-\pi t}\, dt \quad \ldots\ldots \quad (4)$$

(4) We define an ideal set of experimental conditions under which the individual under observation must act. The individual is given an initial sum of money (S), upon which he may draw at will. All money not drawn upon bears interest, compounded at a given rate. Moreover, the individual must so allocate his expenditures that there be no balance left at the end of the

period. Mathematically, we may state this as the imposition of the following isoparametrical side condition on the previous functional to be made a maximum :

$$S = \int_0^b x(t) e^{-rt}\, dt \quad \ldots \quad (5)$$

where r corresponds to the return on the unused balance.

Given these assumptions, it is possible to state certain necessary conditions between the time shape of income expenditure, an observable phenomenon, and the marginal utility of money income. Without knowing the form of the utility function itself, we can state on *a priori* mathematical grounds these relationships. Later from the actual observable shape of the income expenditure as a function of time, we shall be able to deduce the actual shape of the utility function, invariant except for a linear transformation, i.e. scale and origin constants.

In order to simplify the exposition of the necessary conditions of a maximum of (3), subject to the side conditions (4), we may avail ourselves of a Lagrange multiplier λ, and reduce our problem to the equivalent one of maximising the following functional :

$$\int_0^b U(x) e^{-\pi t} dt - \lambda\left[\int_0^b x(t) e^{-rt} dt - S\right] \quad \ldots \quad (4')$$

A necessary condition that this functional be a maximum, may be secured heuristically by disregarding the integral signs, and differentiating with respect to x, treating it as an independent variable. This yields us the following Euler-necessary condition for a maximum : [1]

$$U'(x) e^{-\pi t} - \lambda\, e^{-rt} = 0 \quad \ldots \quad (6)$$

where $U'(x)$ represents the marginal utility of income. We may re-write this as follows :

$$U'(x) = \lambda e^{(\pi - r)t} \quad \ldots \quad (6')$$

Here we have the precise way in which marginal utility will vary through time under the assumptions we have made. In 6 and 6′, λ is a constant depending upon the original amount of money S, and the actual unit in terms of which utility is reckoned. From 6′, we can solve explicitly for x as a function of t providing that we know the form of the utility function. On the other hand, and this is the object of our search, if we know x as a function of t, we can always arrive at the form of the utility function.

Now, by observing the results of our experiment (i.e. the behaviour of the individual in the allocation of his expenditures over time), we arrive at an empirical function between x and t as follows :

$$x = \bar{f}(t) \quad \ldots \quad (7)$$

[1] For the purpose of this paper it does not seem advisable to go into the problem of providing additional necessary conditions or sufficiency conditions. At a later point, where relevant, notice will be taken of another necessary condition. Furthermore, in what follows we should treat all functions as if single-valued. Where this is unjustified, the argument may be easily modified.

where the bar over the function indicates that it is one whose form is known to us. We may re-write equation (7) in the following manner :

$$t = \bar{q}(x) \quad \text{(7')}$$

where $\bar{q}$ is the inverse function of $\bar{f}$ and is known to us.

Now we may substitute 7′ in 6′ and get the following unambiguous expression for our marginal utility function :

$$U'(x) = \lambda e^{(\pi-r)\bar{q}(x)} \quad \text{(8)}$$

By simple integration we may easily come to the form of the utility function as follows :

$$U(x) = \lambda \int_C e^{(\pi-r)\bar{q}(x)} dx \quad \text{(8')}$$

where C indicates the limits of integration to be taken, subject to the condition that this expression be finite.

Our main quest is now over. It remains only to investigate the relationships which our analysis may reveal, and to discuss the serious limitations of our approach. For this purpose it is most useful to consider the following dimensionless elasticity expression between marginal utility and income :

$$E_{u'x} = E = xU''(x)/U'(x) \quad \text{(9)}$$

Differentiating (6′) with respect to x we get

$$U''(x) = (\pi-r)e^{(\pi-r)t}dt/dx \quad \pi \neq r \quad \text{(10)}$$

Therefore, by substitution of (10) in (9),

$$E = x(\pi-r)dt/dx \quad \pi \neq r \quad \text{(11)}$$

In equation (11) we have presented in most convenient form the relations which must hold between our functions in consistence with our assumptions. In general, economists assume on *a priori* grounds that marginal utility decreases with income in a monotonic manner. Moreover, in terms of our previous assumptions we can definitely state as a secondary necessary condition for a relative maximum, that marginal utility be *not* an increasing function of income in a neighbourhood of the equilibrium expenditures at each instant of time. Hence, x being of necessity a positive number we have the following condition :

$$(\pi-r)dt/dx < 0 \quad \pi \neq r \quad \text{(12)}$$

Thus, each of the terms in (12) must be of opposite signs. We may say then that whenever the individual's subjective rate of utility discount is greater (less) than the rate of interest at which he can borrow or lend unlimited amounts, he will allocate his income over some finite period of time in a decreasing (increasing) function of time. This, of course, is consistent with our common sense, intuitional judgments. In the case that these two rates are equal we know immediately from (6′) that marginal utility is a constant throughout time, and hence income must be spent at a constant rate. Here our experiment gives us but one point on the marginal utility function. In order to arrive at its form over an interval we must introduce another interest rate. This suggests a possible way to determine experimentally the value of π, i.e.

by determining for which value of r the rate of money expenditure is a constant.

It may be well to point out that when the interest rate equals the rate of time discount the individual will *not* be conserving the value of his capital assets, in the sense that net saving will be zero. Rather will he be eating up his capital in such a way that it will provide him a steady income over the finite period under consideration (life of the individual, lifespan plus lifespan of immediate heirs, etc.). Only if the individual is maximising such a function over an infinite period of time, i.e. when the rate of amortisation of capital value is zero, will his capital assets remain constant in value in the sense that net savings (investment) are equal to zero.

In the case where the " individual " is maximising such an integral over an infinite period of time, the interest rate being conceived of as remaining constant, we may take the initial amount of money possessed by the " individual " as the discounted value of all the future net yields of all the services (property or human) to which the " individual " has legal claim (uncertainty being assumed absent). In this abstract case, the " individual " would always be increasing the net value of his assets, i.e. be always saving, so long as the interest rate exceeds his rate of discount of utility. The opposite proposition may be formulated for the case in which the rate of utility discount exceeds the perpetual rate of interest. We assume here that the interest rate is taken as a datum by the individual.[1]

Our task now is to indicate briefly the serious limitations of the previous kind of analysis, which almost certainly vitiate it even from a theoretical point of view. In the first place, it is completely arbitrary to assume that the individual behaves so as to maximise an integral of the form envisaged in (2). This involves the assumption that at every instant of time the individual's satisfaction depends only upon the consumption at that time, and that, furthermore, the individual tries to maximise the sum of instantaneous satisfactions reduced to some comparable base by time discount. As has been suggested,[2] we might assume that the individual maximises an integral which contains not only consumption per unit time but also the rate of change of consumption per unit time, and higher derivatives. This is more general in the sense that it includes our assumption as a special case, but still would seem arbitrary. A more general formulation of the problem might conceive of the individual's behaving so as to maximise a *functional* of various time shapes of consumption through time. Mathematically, we might conceive of a *functional* utility index as follows:

$$J = \overset{b}{\underset{a}{Z}}(x\,t) \quad \ldots\ldots \quad (13)$$

or, slightly more generally:

$$F = F(J) \quad \ldots\ldots \quad (14)$$

[1] It may be well to note here the brilliant article by the late F. P. Ramsey on kindred subjects: " A Mathematical Theory of Savings," *Economic Journal*, 1928.

[2] Gerhard Tintner: " A Note on Distribution of Income Over Time," *Econometrica*, V, IV, pp. 60–66.

where $F(J)$ indicates any monotonic transformation of J. This would include as a special case the integrals studied in the Calculus of Variations.[1]

Moreover, and this is the important theoretical implication, there is almost nothing that we can say concerning such a functional on *a priori* grounds. Since we could only know it by its effects, there would seem to be little value in the concept, since it can hardly be argued that it would serve as a convenient construction in simplifying our analysis in view of the inaccessability of such theory to students without an advanced knowledge of mathematics.

A less important point to be noted is the fact that our equations hold only for an individual who is deciding at the beginning of the period how he will allocate his expenditures over the period. Actually, however, as the individual moves along in time there is a sort of perspective phenomenon in that his view of the future in relation to his instantaneous time position remains invariant, rather than his evaluation of any particular year (e.g. 1940). This relativity effect is expressed in the behaviour of men who make irrevocable trusts, in the taking out of life insurance as a compulsory savings measure, etc. The particular results we have reached are not subject to criticism on this score, having been carefully selected so as to take care of this provision. Contemplation of our particular equations will reveal that the results are unchanged even if the individual always discounts from the existing point of time rather than from the beginning of the period. He will still make at each instant the same decision with respect to expenditure as he would have, if at the beginning of the period he were to decide on his expenditure for the whole period. But the fact that this is so is in itself, a presumption that individuals do not behave in terms of our functions.

Moreover, in the analysis of the supply of savings, it is extremely doubtful whether we can learn much from considering such an economic man, whose tastes remain unchanged, who seeks to maximise some functional of consumption alone, in a perfect world, where all things are certain and synchronised. For in any case such a functional would have to be dependent upon certain parameters which are socially determined; "effective" desire for social prestige, length of human life, life cycle of individual economic activity, corporate structure, institutional banking and investment structure, etc. In general, there is strong reason to believe that changes in such parameters are not of an equilibrating nature. Even to generalise concerning these can only be done in terms of a theory of "history" (in itself almost a contradiction in terms). In any case, this would seem to lie in the region which Marshall termed Economic Biology, where the powerful tools of mathematical abstraction will little serve our turn, and direct study of such institutional data would seem in order.

In what way have our assumptions enabled us to arrive at a particular measure of utility rather than merely one index of utility? Reflection as to

[1] If functional theory is to be applied to this problem, it would seem preferable to confine the discussion, at least in the preliminary stages, to functionals which have functional derivatives at every point, i.e. functionals of continuity zero. Cf. Volterra: *Theory of Functionals*. This would seem preferable on economic and mathematical grounds. Such procedure would exclude the integrals discussed by Tintner (ibid.), but would, on the other hand, be much wider than the kind of integral depicted in (2).

the meaning of our Assumption Two, that the individual seeks to maximise an integral of the kind envisaged in (2), will reveal that the individual must make preferences in the Utility dimension itself, that is to say, we must invoke Pareto's Postulate Two, which relates to the possibility of ordering *differences* in utility by the individual.[1] The advantage of our experiment is that it compels the individual to make just such judgments. Thus, with postulates one and two being fulfilled, it is to be expected that utility is uniquely measurable.[2]

In conclusion, any connection between utility as discussed here and any welfare concept is disavowed. The idea that the results of such a statistical investigation could have any influence upon ethical judgments of policy is one which deserves the impatience of modern economists.

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[1] Cf. O. Lange: "The Determinateness of the Utility Function," REVIEW OF ECONOMIC STUDIES, V. I, pp. 218, 225.

[2] Professor Leontief has pointed out to me that this method may be compared with the usual method of deriving demand or supply curves from knowledge of the shifts of these functions through time. Here, it was assumed that we know the shifts of the utility function through time (rate of time discount). This, together with our side conditions and maximisation conditions, enables us to go back from our observed income expenditure function of time, to the utility function itself.